Classical Yang-Mills Vacua on T^3 : Explicit Constructions.

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Abstract

Flat connections for unitary gauge groups on a 3–torus with twisted boundary conditions as well as recently discovered periodic nontrivial flat connections with "nondiagonalizable" triples of holonomies for higher orthogonal and exceptional groups are constructed explicitly in terms of Jacobi theta functions with rational characteristics. The (fractional) Chern-Simons numbers of these vacuum gauge field configurations are verified by direct computation.

1 Introduction

Gauge theories on a torus have been studied for a long time [1]. The main reason why this subject is interesting is that the torus provides a natural infrared cutoff which does not break translational invariance and supersymmetry. The Euclidean 4–dimensional torus is used in lattice calculations. If we stay in the Hamiltonian framework, only space is compactified and the theory is defined on $T^3 \times R$. For small spatial tori the effective coupling constant is also small and the vacuum structure of the theory can be studied in the Born–Oppenheimer framework. This approach is especially useful in the supersymmetric case, where the number of exact vacuum states does not depend on the size of the torus [2]. It was also suggested that the modes

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which are most relevant in the strong coupling regime are topologically distinguished on T^3 [1] .

One of the important technical aspects is that gauge theories on a torus admit nontrivial *twisted* boundary conditions [1]. [1] They have the form

$$
\begin{aligned}
A_i(x+1,y,z) &= PA_i(x,y,z)P^{-1} \,, \\
A_i(x,y+1,z) &= QA_i(x,y,z)Q^{-1} \,, \\
A_i(x,y,z+1) &= SA_i(x,y,z)S^{-1} \,,
\end{aligned}
\tag{1}
$$

where $i = 1,2,3$ and the periods of the torus are normalized to 1. Now, P , Q , and S are constant elements of the gauge group forming so called Heisenberg pairs:

$$
QP = \omega_1 PQ \,, \quad QS = \omega_2 SQ \,, \quad SP = \omega_3 PS \,,
\tag{2}
$$

with ω_i belonging to the center of the group. If at least one of ω_i is a nontrivial element of the center, the conditions (1, 2) mean from the mathematical standpoint that our fiber bundle is topologically nontrivial (not reduced to the direct product $G \times T^3$).

To study the vacuum structure of quantum Yang–Mills theory, one has first to understand the structure of classical vacua. The latter are given by gauge field configurations with zero field strength, flat connections in mathematical language. Classification of all flat connections on a 3–torus is an interesting and nontrivial mathematical problem. For unitary groups it was largely solved in Refs.[1,2]. It turns out that any topologically trivial flat connection $A_i(x,y,z)$ can be gauge–transformed to constant commuting A_i. A distinct vacuum is characterized by a set of holonomies $\Omega_i = \exp\{iA_i\}$, with each holonomy lying on the maximal torus of the group.

When twist is allowed, the moduli space of classical vacua contains generically several disconnected components. Consider the case $S = 1$, which implies $\omega_2 = \omega_3 = 1$, and assume that ω_1 is the primitive N-th root of unity $\epsilon = \exp\{2\pi i/N\}$ (or some other element generating the whole center subgroup Z_N). In this case, the moduli space of vacua factorized over all gauge transformations, including the gauge transformations of "instanton nature"

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[1]This is so if the theory does not involve the fields in the fundamental or other representation for which the group of the center acts faithfully. For example, the twisted boundary conditions are not admissible for standard QCD involving quarks.

which change the Chern–Simons (CS) number of a gauge field configuration by an integer, contains just N isolated points. The CS numbers of these isolated vacua are fractional $N_{CS} = p/N$, $p = 0, \ldots, N - 1$. We will call the connections with fractional CS number *interesting*. [2]

If $\omega_1 = \epsilon^k$, where k is a divisor of N, the moduli space contains N/k disconnected components. In each such component the holonomies Ω_i lie on a subtorus of dimension $k - 1$ of the maximal torus . The corresponding CS numbers are multiple integers of k/N.

It can be shown [3] that the boundary conditions (1) with nontrivial S bring about nothing new and the problem is always reduced to one of the cases described above.

For other groups the situation is more complicated, and the problem was solved only recently. It turned out that, for higher orthogonal and exceptional groups, the moduli space of classical vacua involves disconnected components even in the case of trivial twists [4, 5]. (For symplectic groups with trivial twist, there is only one component. This case was analyzed back in Ref.[2].) The complete classification of periodic flat connections for an arbitrary gauge group was constructed in [7, 8]. With the classical vacuum moduli space in hand, also quantum problem can be solved. In all cases, the number of quantum vacuum state in pure $\mathcal{N} = 1$ supersymmetric Yang–Mills theory coincides with the so called dual Coxeter number $h^\vee$ (or just the adjoint Casimir eigenvalue c_V) of the group. [3] The classification of flat connections with nontrivial twist for nonunitary groups was constructed in [9] (for symplectic and orthogonal groups it was pedagogically explained in the last section of recent [10]). Again, the number of quantum vacuum states always coincides with the dual Coxeter number *independently* of the boundary conditions chosen.

In all these more complicated cases, the basic building block used to construct nontrivial flat connections are flat connections for unitary groups with twisted boundary conditions. Let us recall how it is done for periodic connections. Any flat connection is characterized by a triple of commuting holonomies. In contrast to the simple case of unitary groups, such a commut-

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[2] The word "nontrivial" is too broad and vague while "topologically nontrivial" would not be correct since the topologically nontrivial connections of instanton nature are not interesting in the present context, while certain topologically trivial (periodic) connections are.

[3] For theories involving extra matter supermultiplets it may be completely different [6].

ing triple is not always "diagonalizable", i.e. it cannot be always conjugated (gauge transformed) to the maximal torus (to be precise: conjugated to a triple belonging to the maximal torus). A generic such nondiagonalizable commuting triple (an *exceptional* triple as defined in Ref. [7]) is constructed as follows. One chooses the first holonomy of the triple Ω_1 in such a way that its centralizer (a subgroup of the large group G containing all elements of G commuting with Ω_1) is a group whose semi-simple component is a direct product of several $SU(N_i)$ groups factorized over a subgroup of its center $\prod_i Z_{N_i}$. This factorization results in that the fundamental group of the centralizer involves a finite subgroup as a factor (the elements whose centralizer enjoys this property are called *exceptional*) and this allows one to pick up two commuting elements Ω_2, Ω_3 in the centralizer which cannot be conjugated to its maximal torus. Then the triple $\Omega_1, \Omega_2, \Omega_3$ cannot be conjugated to the maximal torus in G.

Consider as a simplest example the group G_2. It involves a unique (up to conjugation) exceptional element whose centralizer is $[SU(2) \times SU(2)]/Z_2$, where the factorization is done over the diagonal subgroup of the center $Z_2 \times Z_2$ of $SU(2) \times SU(2)$ (i.e. the element $(-1, -1)$ of $SU(2) \times SU(2)$ is identified with 1). Then one can choose the elements $\Omega_{2,3}$ in the centralizer in such a way that their liftings in $SU(2) \times SU(2)$ are $\tilde{\Omega}_2 = (P, P)$, $\tilde{\Omega}_3 = (Q, Q)$ (in obvious notations corresponding to the direct product structure) such that P, Q form a Heisenberg pair, $PQ = -QP$, in each $SU(2)$ component. Factorization over the diagonal Z_2 makes the pair Ω_2, Ω_3 commuting. As a noncommuting Heisenberg pair cannot obviously be conjugated to a maximal torus in $SU(2)$, the pair Ω_2, Ω_3 cannot be conjugated to the maximal torus in $[SU(2) \times SU(2)]/Z_2$ and the whole triple cannot be conjugated to the maximal torus in G_2. Note that any two of the holonomies can be conjugated to a maximal torus in G_2. But the corresponding tori for the subsets $\Omega_{1,2}$, $\Omega_{1,3}$, and $\Omega_{2,3}$ are different.

A similar construction works also in all other cases. It is more or less clear that a periodic flat connection based on an exceptional commuting triple of holonomies can be constructed if a flat connection in $SU(N)$, with holonomies $\Omega_{2,3} = P, Q$ such that P, Q form a Heisenberg pair and Ω_1 belonging to the centralizer of P, Q in $SU(N)$, is known. In particular, the CS number of the former coincides with that of the latter and is, generically, fractional. A fractional CS number of new nontrivial vacua is their important property, which allows one to ascribe to the corresponding quantum vacua

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correct fermionic charges and make contact with what is known about the vacuum structure of supersymmetric Yang–Mills theory in large volume [10]

The existence of the twisted gauge field configurations with fractional CS number was known before. Their numerical study was performed in [11]. The main goal of our paper is to present simple explicit analytic formulae for these configurations.

In sect. 2 we construct a flat connection corresponding to holonomies forming the Heisenberg pair in $SU(N)$. The result is expressed in terms of Jacobi Θ functions with rational characteristics. [4] We perform a direct computation of the CS number of the gauge field configuration thus obtained. We confirm that N_{CS} is a multiple integer of k/N for $\omega_1 = \epsilon^k$ with integer N/k .

In sect. 3 we lift the gauge fields associated with the $SU(N)$ Heisenberg pairs to gauge fields with exceptional triples of commuting holonomies in the orthogonal and exceptional groups (a different construction for the flat periodic connections in $Spin(7)$ has been done in [12]). The computation of the CS number is reduced to the previous case. The same can be done for twisted connections in nonunitary groups. We discuss the simplest such case, which is $Sp(4)$.

In the last section we present our conclusions and prospects for future research.

2 Twisted flat connections in $SU(N)$.

Let us first assume that $\omega_1 = \epsilon$. The zero curvature gauge fields can be represented in the form

$$A_i = U^{-1}\partial_i U \ . \tag{3}$$

Following [1, 2], we search for a gauge group matrix $U(x, y, z)$ obeying the boundary conditions:

$$
\begin{aligned}
U(x + 1) &= PU(x)P^{-1} \\
U(y + 1) &= QU(y)Q^{-1} \\
U(z + 1) &= \epsilon U(z) \ ,
\end{aligned}
\tag{4}
$$

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[4]The analogies with the problem of motion of a charged particle in a constant homogeneous magnetic field on a 2D torus are very instructive here.

where the dependence on "irrelevant" variables (y, z in the first line, etc.
) is not displayed. Apparently, the conditions (4) are compatible with the
conditions for the gauge fields in Eq. (1).

We start our construction of the $SU(N)$ gauge field obeying Eqs.(3), (4)
with the ansatz

$$U \;=\; e^{2\pi iz T(x,y)}, \tag{5}$$

where $T(x,y)$ is a Hermitian $su(N)$ matrix conjugated to the matrix [5]

$$T_0 \;=\; \frac{1}{N}\mathrm{diag}(1,\ldots,1,1-N) \tag{6}$$

Apparently, $U|_{z=0} = 1$ and $U|_{z=1} = \epsilon$, so the third condition of Eq. (4) is
satisfied. The other two conditions translate as the following conditions on
$T(x,y)$:

$$T(x+1) = PT(x)P^{-1}$$
$$T(y+1) = QT(y)Q^{-1} \; . \tag{7}$$

It is rather easy to satisfy these conditions for $SU(2)$. If $P = i\sigma_3$ and
$Q = i\sigma_1$ (any Heisenberg pair in $SU(2)$ can be conjugated to this form) , one
can choose

$$T(x,y) \;=\; \frac{1}{2}\,\frac{\sigma_1\cos(\pi x)+\sigma_3\cos(\pi y)+\sigma_2\cos[\pi(x+y)]}{\sqrt{\cos^2(\pi x)+\cos^2(\pi y)+\cos^2[\pi(x+y)]}}\;, \tag{8}$$

where the square root factor is inserted for proper normalization. A similar
in spirit formula was written in Ref.[11]a for the case $P = i\sigma_1$, $Q = i\sigma_2$,
$S = i\sigma_3$. It is difficult, however, to generalize the solution (8) to the case of
higher N.

To solve (7) for arbitrary N, we first notice that the matrices conjugated
to T_0 form $CP^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}$ orbit of $SU(N)$. They are conveniently
parameterized as follows

$$T_{ij}(x,y) \;=\; \frac{1}{N}\delta_{ij} - \psi_i(x,y)\psi_j^\dagger(x,y), \tag{9}$$

where ψ_i is a N-component complex column normalized to unity:

$$\psi^\dagger\psi = 1. \tag{10}$$

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[5]In fact, it is one of the *fundamental coweights*.

Now, ψ is an element of the fundamental representation of $SU(N)$, and the parameterization (9) of the orbit $\frac{SU(N)}{SU(N-1)\times U(1)}$ may be called *fundamentalization*. A traceless Hermitian matrix $T(x,y)$ from Eq. (9) has $2N-2$ real parameters [N complex parameters in the column ψ_i minus 1 real parameter for the normalization Eq. (10) and minus 1 real parameter for the irrelevant common phase of ψ_i in Eq. (9)], which is equal to the dimension of $\frac{SU(N)}{SU(N-1)\times U(1)}$ space.

The boundary condition (7) is reduced to

$$
\begin{aligned}
\psi(x+1) &= e^{i\alpha(x,y)}P\psi(x) \\
\psi(y+1) &= e^{i\beta(x,y)}Q\psi(y) \,,
\end{aligned}
\tag{11}
$$

where real functions $\alpha(x,y)$ and $\beta(x,y)$ should be chosen to compensate the nontrivial commutant (2) of P and Q and to make $\psi(x+1,y+1)$ uniquely defined. The latter self–consistency condition implies

$$
e^{-i\alpha(x,y)}e^{-i\beta(x+1,y)}e^{i\alpha(x,y+1)}e^{i\beta(x,y)} = \omega_1 = \epsilon \,,
\tag{12}
$$

and we make a choice

$$
\alpha(x,y) = \frac{2\pi y}{N}, \quad \beta(x,y) = 0 \,.
\tag{13}
$$

The phases $\alpha(x,y), \beta(x,y)$ can be interpreted as vector potentials $A_{x,y}$ of an auxiliary constant Abelian magnetic field with flux $\Phi = 1/N$ on the 2–torus (and their exponentials are the corresponding abelian holonomies).

Being expressed in words, Eq. (11) means that we need to construct a global section of a $\frac{SU(N)\times U(1)}{Z_N} = U(N)$ bundle over T^2 with C^N as a typical fiber. $e^{i\alpha(x,y)}P$ and $e^{i\beta(x,y)}Q$ are the transition matrices. The first Chern class of the bundle is

$$
c_1 = \frac{1}{2\pi}\int \mathrm{Tr}\{F\} = N\Phi = 1 \,.
\tag{14}
$$

This is a problem which the Jacobi Θ functions with rational characteristics (see Ref. [13] for definitions and notations) are tailor–made for.

It was shown in Ref. [1] that any Heisenberg pair in $U(N)$ satisfying

$QP = \epsilon PQ$ can be conjugated to

$$P = e^{i\delta_P} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \epsilon & 0 & 0 & \dots \\ 0 & 0 & \epsilon^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$Q = e^{i\delta_Q} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots \end{pmatrix}. \tag{15}$$

By choosing some special $\delta_{P,Q}$, one can also write a "canonical" Heisenberg pair for $SU(N)$, but we do not need this because an overall phase of P, Q is not relevant in Eq. (7).

Notice that Q acts on the column ψ by cyclically shifting its elements one step up so that the second condition in Eq. (11) simply fixes all the components ψ_j in terms of ψ_1,

$$\psi_{1+j}(x, y) = \psi_1(x, y + j) \tag{16}$$

and requires thereby periodicity of ψ_1 when y is shifted by N,

$$\psi_1(x, y + N) = \psi_1(x, y) . \tag{17}$$

All other components ψ_j also enjoy this property. In view of Eqs.(16 , 15, 13), the first condition in Eq. (11) is reduced to

$$\psi_1(x + 1, y) = e^{\frac{2\pi i y}{N}} \psi_1(x, y) . \tag{18}$$

The conditions (17, 18) are obviously satisfied upon the choice

$$\psi_1(x, y) = \mathcal{N}(x, y) \sum_{n \in Z} e^{-\pi(n + \frac{y}{N})^2 + 2\pi i x(n + \frac{y}{N})}, \tag{19}$$

where $\mathcal{N}(x, y)$ is a periodical function of x and y with period 1. Other ψ_j are defined via Eq. (16). The factor $\mathcal{N}(x, y)$ should be chosen such that the normalization condition (10) is satisfied . For $\mathcal{N}$ to be well defined we need

to check that ψ_j do not have a common zero. To this end it is convenient to express ψ_j in terms of Jacobi Θ functions. Using the definition of the theta functions $\Theta_{l/N,m/N}(z,\tau)$ with rational characteristics l/N, m/N (see [13]),

$$\Theta_{l/N,m/N}(z,\tau) = \sum_{n \in Z} e^{i\pi\tau(n+\frac{l}{N})^2 + 2\pi i(n+\frac{l}{N})(z+\frac{m}{N})}, \tag{20}$$

one straightforwardly verifies that

$$\psi_j(x,y) = \mathcal{N}(x,y) e^{-\pi(\frac{y}{N})^2 + 2\pi i x \frac{y}{N}} \Theta_{(j-1)/N,0}\left(x + i\frac{y}{N}, i\right). \tag{21}$$

Now, $\Theta_{l/N,m/N}(z,\tau)$ have zeros at $z = (l/N + p + 1/2)\tau + (m/N + q + 1/2)$, $p,q \in Z$, so $\psi_j(x,y)$ have no common zero. The factor $\mathcal{N}$ can thus be determined as

$$\mathcal{N}(x,y) = \frac{e^{\pi(\frac{y}{N})^2}}{\sqrt{\sum_{l=0}^{N-1} |\Theta_{l/N,0}(x + i\frac{y}{N}, i)|^2}}. \tag{22}$$

A digression is in order here. The conditions (17, 18) are the same as the conditions imposed on a charged particle moving on the "large" 2–torus (with $0 \le x \le 1$, $0 \le y \le N$) in an external homogeneous magnetic field $B = 2\pi/N$. It is known that such a problem is self–consistent only if the total flux $\Phi = B\mathcal{A}/(2\pi)$ is integer. In our case, $\Phi_{\text{large torus}} = 1$. Charged particle on a torus in an arbitrary magnetic field with a given flux was considered in [14]. The solution of the problem for homogeneous field allows one to calculate the functional integral in Schwinger model in topologically nontrivial sectors [15].

If going back to the original "small" torus $0 \le x,y \le 1$, we see a system of N charged particles moving in a magnetic field of flux $\Phi_{\text{small torus}} = 1/N$. Fractional fluxes are now admissible because the corresponding wave functions satisfy the *flavor–twisted* boundary conditions (16). Such conditions were studied before in Ref. [16]. In multiflavor Schwinger model they allow for the presence of Euclidean gauge field configurations with fractional instanton number.

Note that, though the boundary conditions in our problem and in the problem of motion in an external magnetic field are identical, the functions (19, 21) *do* not solve the corresponding Schrödinger equation. The normalization condition (10) which we have to satisfy at *every* point on the torus is alien to the Sturm–Liouville settings.

Substituting Eq. (21) into Eqs.(9), (5), and (3), we obtain the twisted connection we were seeking for. All other solutions to the boundary conditions (4) are related to this particular solution under gauge transformations, including transformations of the instanton type.

Let us now compute the CS number of this field. We have

$$N_{CS} = \frac{1}{8\pi^2} \int_{T^3} \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \tag{23}$$

which is normalized so that, on two flat gauge fields related by an instanton, the CS number differs by 1. Since the connection is flat, we actually need to compute the integral

$$N_{CS} = -\frac{1}{8\pi^2} \int d\mathbf{x}\, \mathrm{Tr}\left\{(\partial_x U^{-1}\partial_y U - \partial_y U^{-1}\partial_x U)U^{-1}\partial_z U\right\}. \tag{24}$$

When the spatial manifold is S^3, Eq. (24) defines $\pi_3[SU(N)]$ and is integer. The same holds for the normal untwisted torus. But in the twisted case the situation is different.

To find (24), we substitute there $U(x,y,z)$ in the form (5), (9). Then $U^{-1}\partial_z U = 2\pi i T(x,y)$. To find the factors $\partial_{x,y}U^{-1}$, $\partial_{x,y}U$, it is convenient to represent U and U^{-1} as follows

$$U = e^{\frac{2\pi i z}{N}}[1 + (e^{-2\pi i z} - 1)\Pi]$$
$$U^{-1} = e^{-\frac{2\pi i z}{N}}[1 + (e^{2\pi i z} - 1)\Pi], \tag{25}$$

with $\Pi_{ij} = \psi_i \psi_j^\dagger$, $\Pi^2 = \Pi$. Then Eq. (24) is reduced to

$$N_{CS} = \frac{1}{\pi i} \int dxdydz \sin^2(\pi z)\mathrm{Tr}\left\{[(\partial_x \Pi)(\partial_y \Pi) - (\partial_y \Pi)(\partial_x \Pi)]\Pi\right\} =$$
$$= \frac{1}{2\pi i} \int dxdy[\partial_x(\psi^\dagger \partial_y \psi) - \partial_y(\psi^\dagger \partial_x \psi)] \tag{26}$$

The last integral involves full derivatives and can be readily done using the boundary conditions (11). The result depends only on the "Abelian vector potentials" $\alpha(x,y), \beta(x,y)$ and coincides with the flux of the corresponding auxiliary magnetic field, so

$$N_{CS} = 1/N \tag{27}$$

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in this case. It is clear that U^p gives rise to a configuration with $N_{CS} = p/N$.

We have considered the case when P and Q commute to the primitive root of unity ϵ. The construction generalizes straightforwardly to the case of $QP = e^{2\pi ik/N}PQ$ when k and N are coprime. Along the same lines as above, one easily obtains $U(\mathbf{x})$ satisfying Eq. (4). The corresponding CS number is equal to k/N. Powers of U complemented by instanton shifts will generate all values of N_{CS} that are multiple integers of $1/N$.

Now suppose that N is not prime and k is its divisor: $N = kM$. Take $\omega_1 = \epsilon^k = e^{2\pi ik/N} = e^{2\pi i/M}$. This element generates a subgroup Z_M of the center group Z_N. In this case, there are moduli of the Heisenberg pairs [17, 7]. The centralizer of a pair is now a continuous subgroup of $SU(N)$. There are some special pairs with centralizer $SU(k)$, [6] while the centralizer of a generic pair is $[U(1)]^{k-1}$, the maximal torus T in $SU(k)$. Let us consider the Heisenberg pair P and Q^k, with P and Q defined in Eq. (15). This pair is of generic type, so the moduli space of the corresponding flat connections is $[U(1)]^{k-1}$. In terms of the matrix U in Eq. (3) these moduli show up in the boundary conditions which now read as follows:

$$
\begin{aligned}
U(x+1) &= e^{iT_1}PU(x)P^{-1} \\
U(y+1) &= e^{iT_2}Q^kU(y)Q^{-k} \\
U(z+1) &= \epsilon e^{iT_3}U(z),
\end{aligned}
\tag{28}
$$

where $\exp\{iT_j\}$ lie on the torus T. The moduli T_j are easily taken into account by substituting

$$
U(x) = e^{i\mathbf{x}\mathbf{T}}\tilde{U}(\mathbf{x}) ,
\tag{29}
$$

so that $\tilde{U}(\mathbf{x})$ satisfy the boundary conditions (28) without the factors $\exp\{iT_j\}$.

To find $\tilde{U}(\mathbf{x})$, we can use the same ansatz as before [see Eqs. (5), (9), (10)] and everything goes through in a parallel way with the the only change that, to compensate for the commutant of P and Q^k, we should take $\alpha(x, y) =$

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[6]Their explicit form is

$$
P = e^{2\pi il/k}\mathrm{diag}(p, \ldots, p), \qquad Q = e^{2\pi is/k}\mathrm{diag}(q, \ldots, q) ,
$$

$p, q \in SU(M)$, $qp = e^{2\pi i/M}pq$, $l, s = 1, \ldots, k$. Note that the pairs characterized by different l, s are inequivalent to each other by conjugation.

$2\pi iky/N$ so that the flux of the auxiliary magnetic field is now k/N. Eq. (16) now reads

$$\psi_{l+kj}(x,y) \;=\; \psi_l(x,y+j)\;,\; l \;=\; 1,\ldots,k\;;\qquad j=1,\ldots,M-1\;. \qquad (30)$$

It expresses all components of ψ in terms of the first k components. We can safely assume that only one of $\psi_l,\; l=1,\ldots,k$, say, the one with $l=1$, is different from zero. The functions $\tilde{U}$ obtained with more general assumptions obey the same boundary conditions and thus are all gauge equivalent. The conditions (30) imply that $\psi_1(x,y)$ and all its "descendants" are periodic in y with period M. On the other hand,

$$\psi_1(x+1,y) \;=\; e^{2\pi iy/M}\psi_1(x,y)\;. \qquad (31)$$

The conditions (30), (31) have exactly the same form as Eqs. (16), (18), only N is substituted by M. A set of functions ψ satisfying these conditions can be chosen as

$$\psi_{1+kj}(x,y) \;=\; \mathcal{N}(x,y)e^{-\pi(\frac{y}{M})^2+2\pi ix\frac{y}{M}}\Theta_{j/M,0}\left(x+i\frac{y}{M},i\right) \qquad (32)$$

with $j=0,\ldots,M-1$. The CS number is computed as easy as before and is equal to 1/M. The connections with $U^{(p)}(x,y) = [\tilde{U}(x,y)]^p$ (they satisfy the boundary conditions (28) with $T_i = 0$ and $\epsilon \to \epsilon^p$) have the Chern–Simons number p/M.

3   Flat connections for exceptional triples.

As was explained in the Introduction, exceptional triples in higher orthogonal and exceptional groups are intimately connected with Heisenberg pairs for unitary groups. We will explain now how the corresponding periodic connections are build up in terms of the twisted connections for $SU(N)$.

Let G be a simple connected simply connected Lie group. Pick up a generic exceptional Ω_1 whose centralizer is a product $H = SU(N_1) \times SU(N_2) \times \ldots$ factorized over a subgroup of its center. The liftings of $\Omega_{2,3}$ in H are Heisenberg pairs in each component $SU(N_i)$. Let us find out how Ω_1 is embedded into H. A priori, Ω_1 could be an element of the center $Z_{N_1} \times Z_{N_2} \times \ldots$ of H. In fact, the known explicit form of Ω_1 allows one to

conclude that it is unity (the trivial element of the center) in all $SU(N_j)$ components but the component $SU_\theta(N_\theta)$ which contains the coroot $\theta^\vee$ corresponding to the highest root θ as a generator in its maximal torus.

Indeed, as follows from Theorem 1 of Ref.[7], any exceptional element can be conjugated to

$$\Omega_1 = \exp\left\{2\pi i \sum_{j=1}^{r} s_j \omega_j\right\}, \tag{33}$$

where the sum runs over all nodes of the Dynkin diagram of the corresponding group, r is the rank of the group, ω_j are the fundamental coweights, i.e. the elements of the Cartan subalgebra commuting with all simple root vectors but one, $[\omega_j, T_{\alpha_k}] = \delta_{jk} T_{\alpha_k}$, and the real numbers s_i (so called *Kac coordinates*) have the following properties:

- $s_j \geq 0$.

- $\sum_{j=1}^{r} a_j s_j = 1$, where a_j are *Dynkin labels*, or the integer coefficients of expansion of the highest root θ over simple roots, $\theta = \sum_j a_j \alpha_j$, of the corresponding node.

- The greatest common divisor m of all *dual* Dynkin labels $a_j^\vee = a_j \langle \alpha_j, \alpha_j \rangle / 2$ dwelling on the nodes with $s_j \neq 0$ is nontrivial $m > 1$. The dual Dynkin labels are the coefficients of expansion of the coroot $\theta^\vee$ corresponding to the highest root θ over the simple coroots $\alpha^\vee$. For simply laced groups $a_j^\vee = a_j$. The integer m is an important characteristic of the corresponding exceptional triple and can be called its *order*.

To find the (semi–simple part of) the centralizer, one has to *(i)* Consider the extended Dynkin diagram of the group including the simple roots α_j and the root $\alpha_0 = -\theta$ and to cross out all the nodes for which $s_j \neq 0$. Generically, we are left with a product H of unitary groups. *(ii)* Factorize it over Z_m embedded in a certain way in the center of H.

Now, ω_j entering the sum in Eq. (33) commute with all root vectors corresponding to the nodes not entering the sum. The relation $[\omega_j, T_{\alpha_0}] = -a_j T_{\alpha_0}$ holds. Taking into account this and the condition $\sum_j a_j s_j = 1$, we see that $g = \sum_j s_j \omega_j$ commutes with all components $SU(N_i)$ not involving the root α_0 and that $[g, T_{\alpha_0}] = -T_{\alpha_0}$. Therefore, g represents (up to irrelevant

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sign) the fundamental coweight ω_0 associated with the root α_0 in $SU_\theta(N_\theta)$. This coweight is conjugate to the matrix T_0 in Eq. (6) and one easily sees that the element (33) is trivial from the viewpoint of all subgroups $SU(N_j) \subset H$ but the group $SU_\theta(N_\theta)$, where it represents a generating element of the center. [7] By inspection of different cases [7], one can observe that N_θ always coincides with the order m of the triple.

Let now $\tilde{\Omega}_j$ stand for Ω_j projected onto $SU_\theta(m)$. Take the matrix $U(x, y, z)$ obeying the boundary conditions

$$
\begin{aligned}
U(x + 1) &= \tilde{\Omega}_2 U(x) \tilde{\Omega}_2^{-1} \\
U(y + 1) &= \tilde{\Omega}_3 U(y) \tilde{\Omega}_3^{-1} \\
U(z + 1) &= \tilde{\Omega}_1 U(z).
\end{aligned}
\tag{34}
$$

Take the explicit expression for this matrix from the previous section and lift it up to the group G. The last subtlety is that the gauge field corresponding to such lifting U_G is not periodic. Instead,

$$
\begin{aligned}
A_i(x + 1) &= \Omega_2 A_i(x) \Omega_2^{-1} \ , \\
A_i(y + 1) &= \Omega_3 A_i(y) \Omega_3^{-1} \ , \\
A_i(z + 1) &= A_i(z) \ .
\end{aligned}
\tag{35}
$$

However, this is curable, since Ω_2 and Ω_3 can be simultaneously conjugated to the maximal torus in G,

$$
\Omega_2 = e^{ia}, \ \Omega_3 = e^{ib}, \ [a, b] = 0 \ .
\tag{36}
$$

Thus, taking instead of U_G the element $\tilde{U}_G$

$$
\tilde{U}_G(x, y, z) = U_G(x, y, z) e^{i(ax + by)},
\tag{37}
$$

one finally obtains the periodical zero curvature gauge field corresponding to the exceptional triple of holonomies Ω_1, Ω_2, Ω_3.

---

[7]To be quite precise, one should not say " Ω_1 *is* trivial, etc ", but rather that it *can be chosen* as such. Indeed, the true centralizer is not H, but H factorized over a nontrivial subgroup of its center, and different liftings of Ω_i in H are possible. The final result does not depend, of course, on the choice of the lifting. See the discussion of the $Spin(7)$ example below.

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Let us now comment on the CS number of the flat connection obtained in this way. First of all, one can verify by direct computation that CS does not change under the transformation (37), so that one can work with U_G, not with $\tilde{U}_G$. The total CS is equal to the sum of the CS numbers in each component $SU(N_i) \subset H$ weighted with certain integer factors n_i reflecting a particular way of how $SU(N_j)$ is embedded in the large group G. For simply laced groups $n_j = 1$ in all cases. For the groups $Spin(2r+1)$ we have generically $H = SU_{\alpha_1}(2) \times SU_{\alpha_2}(2) \times SU_\theta(2)$, where one of the roots $\alpha_{1,2}$ is short. If α_1 is short, $n_1 = 2$ (the corresponding coroot is long and the contribution to the integral in Eq. (23) is proportional to $\langle \alpha_1^\vee, \alpha_1^\vee \rangle = 2\langle \alpha_2^\vee, \alpha_2^\vee \rangle$) and $n_2 = n_\theta = 1$. For G_2, $H = SU_\alpha(2) \times SU_\theta(2)$, where α is the short root with $n_\alpha = 3$, while $n_\theta = 1$. Thus, $n_\theta = 1$ in all cases (actually, it is a theorem that the highest root θ is always long and the corresponding coroot $\theta^\vee$ is always short).

Now, for the components not involving θ, Ω_1 is represented by the unity and it is known [1, 2] that the gauge field configurations constructed via matrices U satisfying the boundary conditions (34) with $\tilde{\Omega}_1 = 1$ have instanton nature and an integer Chern–Simons number, which is as good as zero for our purposes. In fact, one can just take $U = 1$ in all these subgroups and forget about them.

Speaking of the component $SU_\theta(m)$, we have seen that Ω_1 is represented there by a generating element of the center Z_m. The calculation of the CS number boils down to the one from the previous section. N_{CS} depends on the commutant of $\tilde{\Omega}_2$ and $\tilde{\Omega}_3$ and is an integer multiple of $1/m$.

To make things absolutely clear, let us illustrate this general construction in the case of $Spin(7)$ group, the smallest orthogonal group where one meets an exceptional triple.

Consider the extended Dynkin diagram of $Spin(7)$ (Fig. 1). The simple coroots are $\alpha^\vee = e_1 - e_2$, $\beta^\vee = e_2 - e_3$, $\gamma^\vee = 2e_3$, where $e_{1,2,3}$ are the generators of rotation in 3 independent planes. The highest coroot is $\theta^\vee = \alpha^\vee + 2\beta^\vee + \gamma^\vee = e_1 + e_2$ (remember that $a_\gamma^\vee = a_\gamma/2 = 1$) and it coincides in this case with the fundamental coweight ω_β. The first element of the triple is

$$\Omega_1 = e^{i\pi\omega_\beta} = e^{i\pi\theta^\vee} \ . \tag{38}$$

The centralizer of such Ω_1 in $Spin(7)$ is $[SU_\alpha(2) \times SU_\gamma(2) \times SU_\theta(2)]/Z_2$, where Z_2 is the diagonal subgroup in the center group $Z_2^\alpha \times Z_2^\gamma \times Z_2^\theta$. Then
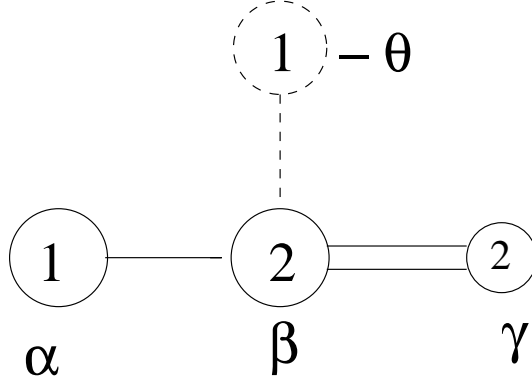
Figure 1: Dynkin diagram for $Spin(7)$ with its Dynkin labels.

the Heisenberg pair in $SU_\alpha(2) \times SU_\gamma(2) \times SU_\theta(2)$ can be taken as follows

$$\Omega_2 = (i\sigma_3, i\sigma_3, i\sigma_3) \,,$$
$$\Omega_3 = (i\sigma_1, i\sigma_1, i\sigma_1) \,, \tag{39}$$

where σ_i are the standard Pauli matrices. The expression (38) is rewritten as $\Omega_1 = (1, 1, -1) \equiv (-1, -1, 1)$ in the direct product notations. Choosing the first possibility, lifting the twisted connection in $SU_\theta(2)$ up to $Spin(7)$, and performing the gauge transformation (37), we obtain an interesting periodic $Spin(7)$ connection with $N_{CS} = 1/2$. (If choosing the second possibility, we obtain the same result up to an irrelevant integer. Indeed, $n_\gamma = 2$ and the twisted connection in the group $SU_\gamma(2)$ provides an integer contribution to the integral (23). The twisted connection of the subgroup $SU_\alpha(2)$ provides the contribution $1/2$ to N_{CS}.)

Consider now the connections whose holonomies form *Heisenberg triples* ("almost commuting" triples by the terminology of Ref. [9]), i.e. triples Ω_i of group elements such that $\Omega_i\Omega_j = c_{ij}\Omega_j\Omega_i$, with c_{ij} being nontrivial elements of the center of the group , for nonunitary groups G. They also can be expressed in terms of twisted unitary group connections constructed in the previous section. We will describe the simplest such example with the group $Sp(4)$ and perform the calculation of Witten index for the $\mathcal{N} = 1$ supersymmetric gauge theory based on this group.

$Sp(4)$ is a subgroup of $SU(4)$ containing all unitary 4×4 matrices U

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satisfying the condition $U^T I U = I$, where I is an antisymmetric symplectic matrix, which can be chosen in the form $I = \text{diag}(i\sigma_2, i\sigma_2) \equiv \mathbf{1} \otimes i\sigma_2$. The group $Sp(4)$ has center Z_2 with nontrivial element $\text{diag}(-\mathbf{1}, -\mathbf{1}) \equiv -\mathbf{1} \otimes \mathbf{1}$.

Heisenberg triples are constructed by studying first the Heisenberg pairs in $Sp(4)$ [9]. Any pair P, Q with $QP = -PQ$ can be conjugated to

$$P = e^{i\alpha\sigma_2} \otimes i\sigma_3, \qquad Q = e^{i\beta\sigma_2} \otimes i\sigma_\mathbf{1} , \tag{40}$$

where the angular variables α, β are moduli. The centralizer of a generic pair P, Q in Eq. (40) is $U(1)$, the corresponding matrices being given by

$$S = e^{i\gamma\sigma_2} \otimes \mathbf{1} . \tag{41}$$

The Heisenberg triples P, Q, S give rise to the "principal" component (i.e. the component involving the perturbative vacuum $A_i = 0$) of the moduli space of classical vacua. They all have zero (or integer) Chern–Simons number. As the rank of the centralizer of the pair P, Q is 1, we obtain $r + 1 = 2$ quantum vacuum states.

There are, however, some special points in the moduli space of the pairs where the centralizer is larger. There are three points: $\alpha = 0, \beta = \pi/2$, $\alpha = \pi/2, \beta = 0$, and $\alpha = \pi/2, \beta = \pi/2$ where the centralizer is $SU(2)$ [7, 18], but, as $SU(2)$ is connected and contains unity, we are still in the principal vacuum sector. If $\alpha = \beta = 0$, the centralizer is $O(2)$ and involves besides (41) the disconnected component $e^{i\gamma\sigma_2}\sigma_3 \otimes \mathbf{1}$, which is equivalent by conjugation to the element $\sigma_3 \otimes \mathbf{1} = \text{diag}(\mathbf{1}, -\mathbf{1})$ (P, Q are invariant under such conjugation) [9, 10]. Now, the unique up to conjugation exceptional Heisenberg triple of holonomies

$$P = \mathbf{1} \otimes i\sigma_3, \qquad Q = \mathbf{1} \otimes i\sigma_1 , \qquad S = \sigma_3 \otimes \mathbf{1} \tag{42}$$

defines the unique up to conjugation interesting twisted connection in $Sp(4)$, which is further promoted to the supersymmetric quantum vacuum state. All together, we have $2 + 1 = 3$ vacuum states which coincides with the counting $r_{Sp(4)} + 1$ based on the analysis of periodic connections. The interesting connection based on the triple (42) is obtained from a matrix U satisfying

$$\begin{aligned} U(x+1) &= PU(x)P^{-1} \\ U(y+1) &= QU(y)Q^{-1} \\ U(z+1) &= S \cdot U(z) , \end{aligned} \tag{43}$$

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These boundary conditions can be easily satisfied with the ansatz $U(\mathbf{x}) = \mathrm{diag}(\mathbf{1}, u(\mathbf{x}))$, where u is the $SU(2)$ matrix satisfying the twisted boundary conditions (4) and which was found before. The corresponding gauge field configuration has Chern–Simons number $1/2$.

4 Discussion

We have described explicitly classical vacua in Yang-Mills on T^3, that is we have constructed analytic expressions for twisted flat connections in unitary groups and have shown that, by a proper embedding, they define also interesting periodic and also twisted flat connections in more complicated groups. The explicit form of the gauge fields allowed us to verify the corresponding fractional CS charges [9] by direct computation.

With our explicit construction in hand, the next challenge is to find Euclidean 4–dimensional gauge field solutions to the classical equations of motion which interpolate between different vacua in complicated groups.

It is not difficult to present a heuristic reasoning in favor of existence of such solutions on $T^3 \times R$ [5]. Consider a flat connection $A_i^{\mathrm{flat}}(\mathbf{x})$ belonging to a topologically nontrivial component not involving the configuration $A_i = 0$. Multiply it by a function $f(\tau)$ of Euclidean time such that $f(\infty) = 1$ and $f(-\infty) = 0$. The interpolating configuration $f(\tau)A_i(\mathbf{x})$ has a nonzero field strength and a nonzero Euclidean action. The latter is not necessarily minimal, but it is clear that exploring the directions in the functional space which make the action lower, we will finally find a configuration with minimal action, i.e. the classical solution.

Of course, this does not tell us what *is* this solution and, in particular, what is its action. Also, this reasoning does not exclude that the configuration minimizing the action becomes singular (cf. the problem of finding instanton solutions of unit charge on T^4, where such a "singularization" occurs, indeed [19]).

These issues can be clarified, bearing in mind the fact that flat connections for complicated groups are constructed out of twisted flat connections for $SU(N)$. It was proven earlier [20] that Euclidean solutions with fractional instanton number $\nu = 1/N, \ldots$ which interpolate between the perturbative vacuum $A_i = 0$ and a nontrivial twisted vacuum exist. They are not singular and satisfy also self–duality equations $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$, which means that their

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action is equal to $8\pi^2/N$. Such solutions were studied numerically in Refs. [11].

Performing a proper embedding and perhaps a gauge transformation like in Eq. (37), we are led to the existence of self–dual solutions on $T^3 \times R$ that interpolate between flat connections belonging to different topological classes as discussed above for an arbitrary group. The instanton numbers of these solutions are multiple integers of $1/m$, or rather of the differences $1/m - 1/m'$, where m and m' are different admissible orders of exceptional triples. For example, for E_8 the instanton number ν can be as small as $\nu = 1/5 - 1/6 = 1/30$. The action of these solutions is $8\pi^2\nu$ in all cases.

Hopefully, our explicit constructions will help in finding an educated guess about analytic form of those solutions. Another interesting problem is to study possible dualities relating the classical vacua with different gauge groups and different boundary conditions. Given the explicit formulae containing Θ functions with rational characteristics, one could expect to see many such dualities associated with the duality relations between Θ functions.

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